THE BINOMIAL THEOREM

We use the **Binomial Theorem** to raise a binomial to the n^{th} power where n is a positive integer. Let's begin by looking for patterns.

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Let's look at the patterns that exist in $(a + b)^n$.

1. Each expansion has n + 1 terms.
2. The powers of a decrease by 1 each term from n to 0.
3. The powers of b increase by 1 each term from 0 to n.
4. The sum of the powers in each term is n.

To see another pattern, let's look at the coefficients:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

This array is called **Pascal's Triangle**. Each entry is the sum of the pair of numbers immediately above it. For example, the 20 in the last row is the sum of the 10 and 10 immediately above it.

Example 1 Expand $(x + y)^4$

Solution: The coefficients of this expansion are 1 4 6 4 1. Following the other patterns, we get:

$$(x + y)^4 = 1 \cdot x^4 + 4 \cdot x^3 \cdot y + 6 \cdot x^2 \cdot y^2 + 4 \cdot x \cdot y^3 + 1 \cdot y^4$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

Example 2 Expand $(2x + 3)^4$

Solution: The coefficients of this expansion are 1 4 6 4 1. Following the other patterns, we get:

$$\begin{aligned}(2x + 3)^4 &= 1(2x)^4 + 4(2x)^3(3) + 6(2x)^2(3)^2 + 4(2x)(3)^3 + 1(3)^4 \\ &= 1(16x^4) + 4(8x^3)(3) + 6(4x^2)(9) + 4(2x)(27) + 1(81) \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Example 3 Expand $(2x – 1)^5$.

Solution: The coefficients of this expansion are 1 5 10 10 5 1. Following the other patterns, we get the following:

$$\begin{aligned}(2x - 1)^5 &= 1(2x)^5 + 5(2x)^4 (-1) + 10(2x)^3 (-1)^2 + 10(2x)^2 (-1)^3 + 5(2x)(-1)^4 + 1(-1)^5 \\ &= 1(32x^5) + 5(16x^4)(-1) + 10(8x^3)(1) + 10(4x^2)(-1) + 5(2x)(1) + 1(-1) \\ &= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1\end{aligned}$$

Note: Notice that the signs alternate +, -, +, -, . . .

Exercise Set

Expand each of the following using the Binomial Theorem and then simplify each expression.

1. $(x + 2)^4$
2. $(x + 1)^5$
3. $(x - 2y)^5$
4. $(x - 3y)^4$
5. $(3x + y)^3$
6. $(4x + y)^3$
7. $(x - y)^6$
8. $(x - 1)^6$
9. $(2x - 3)^4$
10. $(3x - 2)^4$

<u>Answers to Odd Problems</u>

1. $x^4 + 8x^3 + 24x^2 + 32x + 16$
3. $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$
5. $27x^3 + 27x^2y + 9xy^2 + y^3$
7. $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$
9. $16x^4 - 96x^3 + 216x^2 - 216x + 81$